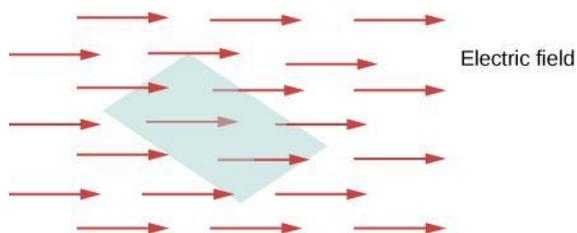


Electric Flux

The concept of flux describes how much of something goes through a given area. More formally, it is the dot product of a vector field (in this chapter, the electric field) with an area. You may conceptualize the flux of an electric field as a measure of the number of electric field lines passing through an area. The larger the area, the more field lines go through it and, hence, the greater the flux; similarly, the stronger the electric field is (represented by a greater density of lines), the greater the flux. On the other hand, if the area rotated so that the plane is aligned with the field lines, none will pass through and there will be no flux.



The flux of an electric field through the shaded area captures information about the “number” of electric field lines passing through the area. The numerical value of the electric flux depends on the magnitudes of the electric field and the area, as well as the relative orientation of the area with respect to the direction of the electric field.

A macroscopic analogy that might help you imagine this is to put a hula hoop in a flowing river. As you change the angle of the hoop relative to the direction of the current, more or less of the flow will go through the hoop. Similarly, the amount of flow through the hoop depends on the strength of the current and the size of the hoop. Again, flux is a general concept; we can also use it to describe the amount of sunlight hitting a solar panel or the amount of energy a telescope receives from a distant star, for example.

To quantify this idea, figure (a) shows a planar surface S_1 of area A_1 that is perpendicular to the uniform electric field $\vec{E} = E\hat{y}$. If N field lines pass through S_1 , then we know from the definition of electric field lines that $N/A_1 \propto E$, or $N \propto EA_1$.

The quantity EA_1 is the electric flux through S_1 . We represent the electric flux through an open surface like S_1 by the symbol Φ . Electric flux is a scalar quantity and has an SI unit of newton-meters squared per coulomb ($N \cdot m^2/C$). Notice that $N \propto EA_1$ may also be written as $N \propto \Phi$, demonstrating that *electric flux is a measure of the number of field lines crossing a surface*.

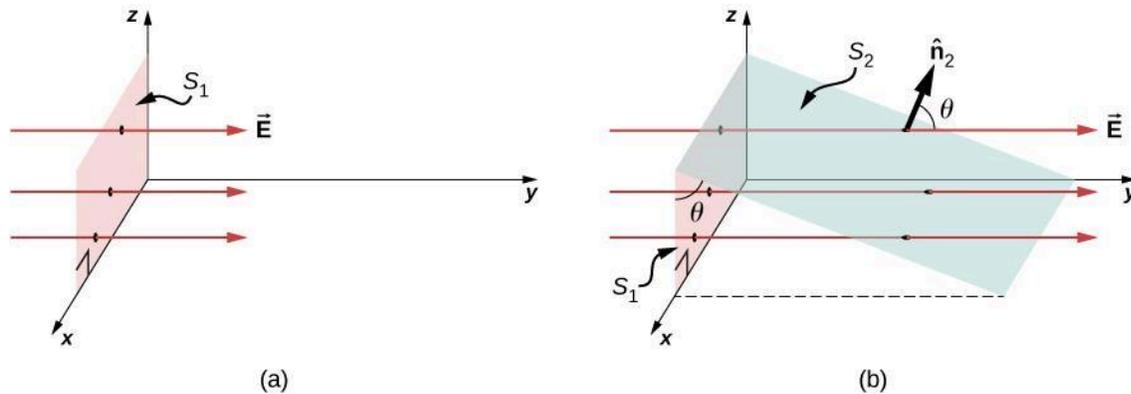


Figure (a) A planar surface S_1 of area A_1 is perpendicular to the electric field \vec{E} . N field lines cross surface S_1 . (b) A surface S_2 of area A_2 whose projection onto the xz -plane is S_1 . The same number of field lines cross each surface.

Now consider a planar surface that is not perpendicular to the field. How would we represent the electric flux? The figure above (b) shows a surface S_2 of area A_2 that is inclined at an angle θ to the xz -plane and whose projection in that plane is S_1 (area A_1). The areas are related by $A_2 \cos \theta = A_1$. Because the same number of field lines crosses both S_1 and S_2 , the fluxes through both surfaces must be the same. The flux through S_2 is therefore $\Phi = EA_1 = EA_2 \cos \theta$. Designating \hat{n}_2 as a unit vector normal to S_2 , we obtain

$$\Phi = \vec{E} \cdot \hat{n}_2 A_2.$$

Interactive

Check out this video(<https://www.youtube.com/watch?v=xsN9zDHRcA>) to observe what happens to the flux as the area changes in size and angle, or the electric field changes in strength.

Area Vector

For discussing the flux of a vector field, it is helpful to introduce an area vector \vec{A} . This allows us to write the last equation in a more compact form. What should the magnitude of the area vector be? What should the direction of the area vector be? What are the implications of how you answer the previous question?

The area vector of a flat surface of area A has the following magnitude and direction:

- Magnitude is equal to area (A)
- Direction is along the normal to the surface (\hat{n}); that is, perpendicular to the surface.

Since the normal to a flat surface can point in either direction from the surface, the direction of the area vector of an open surface needs to be chosen, as shown in the figure below.

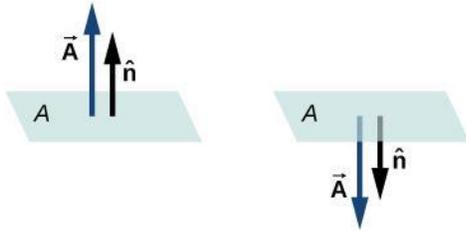


Figure The direction of the area vector of an open surface needs to be chosen; it could be either of the two cases displayed here. The area vector of a part of a closed surface is defined to point from the inside of the closed space to the outside. This rule gives a unique direction.

Since \hat{n} is a unit normal to a surface, it has two possible directions at every point on that surface. For an open surface, we can use either direction, as long as we are consistent over the entire surface. Part (c) of the figure shows several cases.

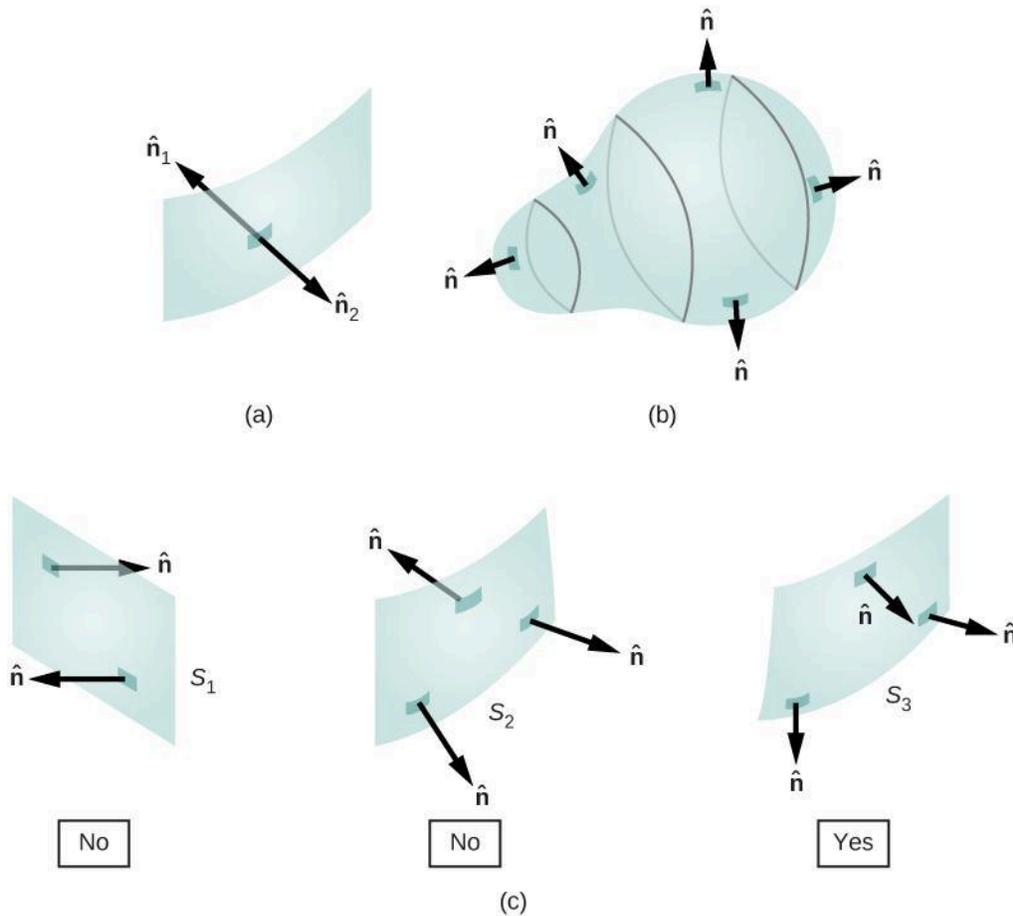


Figure (a) Two potential normal vectors arise at every point on a surface. (b) The outward normal is used to calculate the flux through a closed surface. (c) Only S_3 has been given a consistent set of normal vectors that allows us to define the flux through the surface.

However, if a surface is closed, then the surface encloses a volume. In that case, the direction of the normal vector at any point on the surface points from the inside to the outside. On a *closed surface* such as that of (b), \hat{n} is chosen to be the *outward normal* at every point, to be consistent with the sign convention for electric charge.

Electric Flux

Now that we have defined the area vector of a surface, we can define the electric flux of a uniform electric field through a flat area as the scalar product of the electric field and the area vector, as defined in [Products of Vectors](#):

$$\Phi = \vec{E} \cdot \vec{A} \text{ (uniform } \vec{E}, \text{ flat surface).}$$

The figure below shows the electric field of an oppositely charged, parallel-plate system and an imaginary box between the plates. The electric field between the plates is uniform and points from the positive plate toward the negative plate. A calculation of the flux of this field through various faces of the box shows that the net flux through the box is zero. Why does the flux cancel out here?

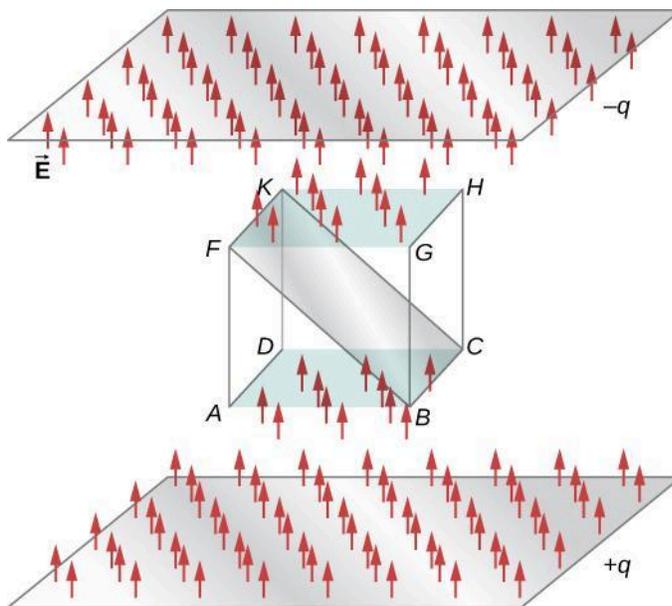


Figure Electric flux through a cube, placed between two charged plates. Electric flux through the bottom face ($ABCD$) is negative, because \vec{E} is in the opposite direction to the normal to the surface. The electric flux through the top face ($EFGH$) is positive, because the electric field and the normal are in the same direction. The electric flux through the other faces is zero, since the electric field is perpendicular to the normal vectors of those faces. The net electric flux through the cube is the sum of fluxes through the six faces. Here, the

net flux through the cube is equal to zero. The magnitude of the flux through rectangle $BCKF$ is equal to the magnitudes of the flux through both the top and bottom faces.

The reason is that the sources of the electric field are outside the box. Therefore, if any electric field line enters the volume of the box, it must also exit somewhere on the surface because there is no charge inside for the lines to land on. Therefore, quite generally, electric flux through a closed surface is zero if there are no sources of electric field, whether positive or negative charges, inside the enclosed volume. In general, when field lines leave (or “flow out of”) a closed surface, Φ is positive; when they enter (or “flow into”) the surface, Φ is negative.

Any smooth, non-flat surface can be replaced by a collection of tiny, approximately flat surfaces, as shown in the figure below. If we divide a surface S into small patches, then we notice that, as the patches become smaller, they can be approximated by flat surfaces. This is similar to the way we treat the surface of Earth as locally flat, even though we know that globally, it is approximately spherical.

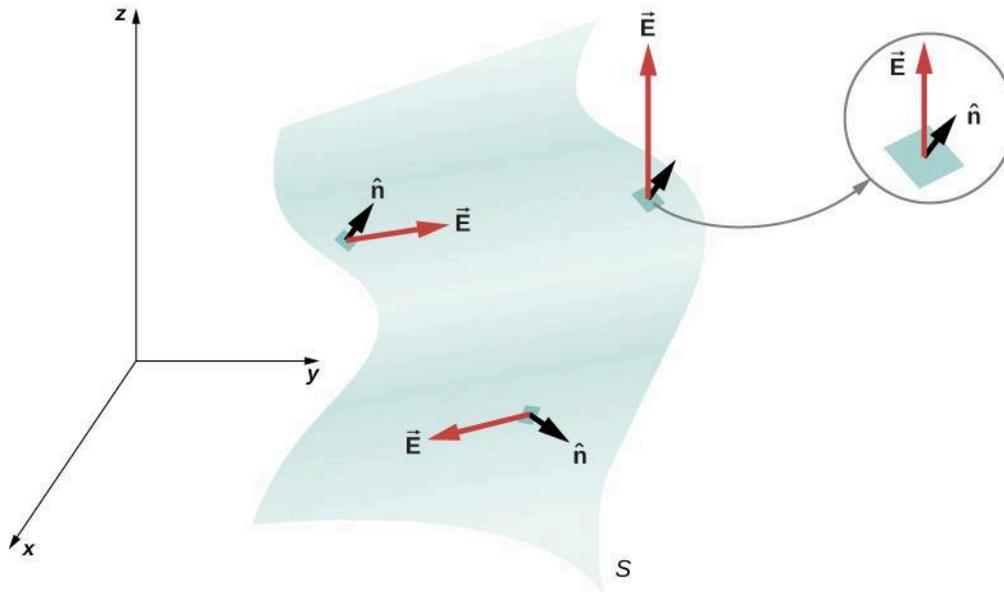


Figure A surface is divided into patches to find the flux.

To keep track of the patches, we can number them from 1 through N . Now, we define the area vector for each patch as the area of the patch pointed in the direction of the normal. Let us denote the area vector for the i th patch by δA_i^{\rightarrow} . (We have used the symbol δ to remind us that the area is of an arbitrarily small patch.) With sufficiently small patches, we may approximate the electric field over any given patch as uniform. Let us denote the average electric field at the location of the i th patch by E_i^{\rightarrow} .

$$E_i^{\rightarrow} = \text{average electric field over the } i\text{th patch.}$$

Therefore, we can write the electric flux Φ_i through the area of the i th patch as

$$\Phi_i = \vec{E}_i \cdot \delta\vec{A}_i \text{ (ith patch)}.$$

The flux through each of the individual patches can be constructed in this manner and then added to give us an estimate of the net flux through the entire surface S , which we denote simply as Φ .

$$\Phi = \sum_{i=1}^N \Phi_i = \sum_{i=1}^N \vec{E}_i \cdot \delta\vec{A}_i \text{ (N patch estimate)}.$$

This estimate of the flux gets better as we decrease the size of the patches. However, when you use smaller patches, you need more of them to cover the same surface. In the limit of infinitesimally small patches, they may be considered to have area dA and unit normal \hat{n} . Since the elements are infinitesimal, they may be assumed to be planar, and \vec{E}_i may be taken as constant over any element. Then the flux $d\Phi$ through an area dA is given by $d\Phi = \vec{E} \cdot \hat{n} dA$. It is positive when the angle between \vec{E}_i and \hat{n} is less than 90° and negative when the angle is greater than 90° . The net flux is the sum of the infinitesimal flux elements over the entire surface. With infinitesimally small patches, you need infinitely

many patches, and the limit of the sum becomes a surface integral. With \int_S representing the integral over S ,

$$\Phi = \int_S \vec{E} \cdot \hat{n} dA = \int_S \vec{E} \cdot d\vec{A} \text{ (open surface)}.$$

In practical terms, surface integrals are computed by taking the antiderivatives of both dimensions defining the area, with the edges of the surface in question being the bounds of the integral.

To distinguish between the flux through an open surface like that of [Figure 6.4](#) and the flux through a closed surface (one that completely bounds some volume), we represent flux through a closed surface by

$$\Phi = \oint_S \vec{E} \cdot \hat{n} dA = \oint_S \vec{E} \cdot d\vec{A} \text{ (closed surface)}$$

where the circle through the integral symbol simply means that the surface is closed, and we are integrating over the entire thing. If you only integrate over a portion of a closed surface, that means you are treating a subset of it as an open surface.

Example

Flux of a Uniform Electric Field

A constant electric field of magnitude E_0 points in the direction of the positive z -axis. What is the electric flux through a rectangle with sides a and b in the (a) xy -plane and in the (b) xz -plane?

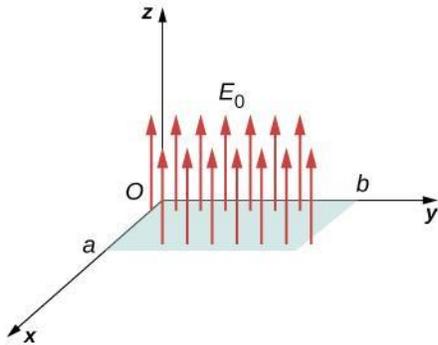


Figure Calculating the flux of E_0 through a rectangular surface.

Strategy

Apply the definition of flux: $\Phi = \vec{E} \cdot \vec{A}$ (uniform \vec{E}), where the definition of dot product is crucial.

Solution

- In this case, $\Phi = \vec{E}_0 \cdot \vec{A} = E_0 A = E_0 ab$.
- Here, the direction of the area vector is either along the positive y -axis or toward the negative y -axis. Therefore, the scalar product of the electric field with the area vector is zero, giving zero flux.

Significance

The relative directions of the electric field and area can cause the flux through the area to be zero.

Example

Flux of a Uniform Electric Field through a Closed Surface

A constant electric field of magnitude E_0 points in the direction of the positive z -axis. What is the net electric flux through a cube?

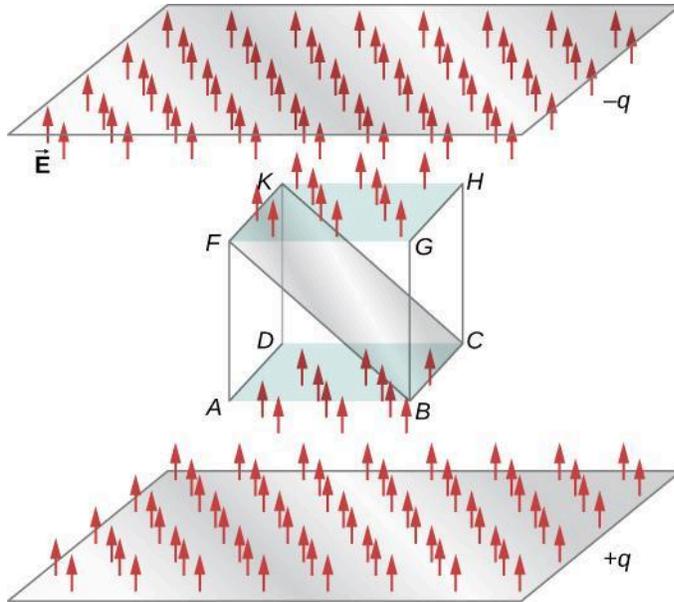


Figure Calculating the flux of E_0 through a closed cubic surface.

Strategy

Apply the definition of flux: $\Phi = \vec{E} \cdot \vec{A}$ (*uniform E*), noting that a closed surface eliminates the ambiguity in the direction of the area vector.

Solution

Through the top face of the cube, $\Phi = E_0 \cdot A = E_0 A$.

Through the bottom face of the cube, $\Phi = E_0 \cdot A = -E_0 A$, because the area vector here points downward.

Along the other four sides, the direction of the area vector is perpendicular to the direction of the electric field. Therefore, the scalar product of the electric field with the area vector is zero, giving zero flux.

The net flux is $\Phi_{net} = E_0 A - E_0 A + 0 + 0 + 0 + 0 = 0$.

Significance

The net flux of a uniform electric field through a closed surface is zero.

Example

Electric Flux through a Plane, Integral Method

A uniform electric field E of magnitude 10 N/C is directed parallel to the yz -plane at 30° above the xy -plane, as shown in the figure below. What is the electric flux through the plane

surface of area 6.0 m^2 located in the xz -plane? Assume that \hat{n} points in the positive y -direction.

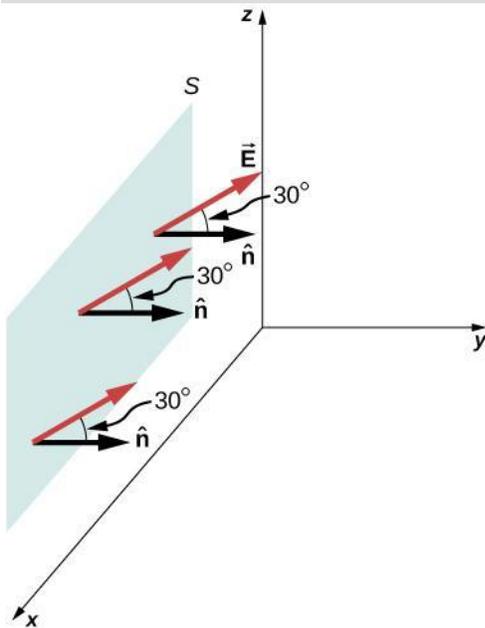


Figure The electric field produces a net electric flux through the surface S .

Strategy

Apply $\Phi = \int_S \vec{E} \cdot \hat{n} dA$, where the direction and magnitude of the electric field are constant.

Solution

The angle between the uniform electric field \vec{E} and the unit normal \hat{n} to the planar surface is 30° . Since both the direction and magnitude are constant, E comes outside the integral. All that is left is a surface integral over dA , which is A . Therefore, using the open-surface equation, we find that the electric flux through the surface is

$$\begin{aligned} \Phi &= \int_S \vec{E} \cdot \hat{n} dA = EA \cos \theta \\ &= (10 \text{ N/C})(6.0 \text{ m}^2)(\cos 30^\circ) = 52 \text{ N} \cdot \text{m}^2/\text{C}. \end{aligned}$$

Significance

Again, the relative directions of the field and the area matter, and the general equation with the integral will simplify to the simple dot product of area and electric field.

Check Your Understanding

What angle should there be between the electric field and the surface shown above in the previous example so that no electric flux passes through the surface?

Example

Inhomogeneous Electric Field

What is the total flux of the electric field $\vec{E} = cy^2\hat{k}$ through the rectangular surface shown in the figure?

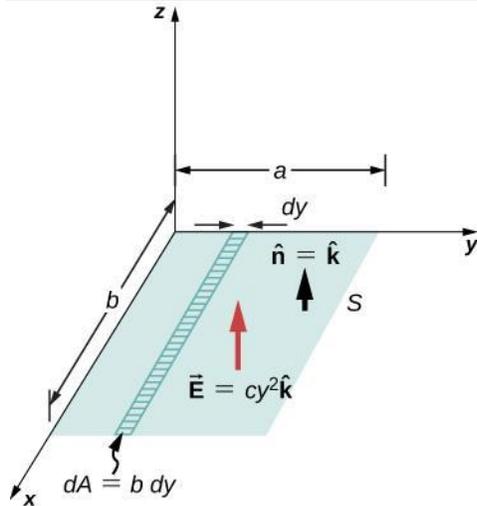


Figure Since the electric field is not constant over the surface, an integration is necessary to determine the flux.

Strategy

Apply $\Phi = \int_S \vec{E} \cdot \hat{n} dA$. We assume that the unit normal \hat{n} to the given surface points in the positive z -direction, so $\hat{n} = \hat{k}$. Since the electric field is not uniform over the surface, it is necessary to divide the surface into infinitesimal strips along which \vec{E} is essentially constant. As shown in the figure above, these strips are parallel to the x -axis, and each strip has an area $dA = b dy$.

Solution

From the open surface integral, we find that the net flux through the rectangular surface is

$$\begin{aligned}\Phi &= \int_S \vec{E} \cdot \hat{n} dA = \int_0^a (cy^2\hat{k}) \cdot \hat{k} (b dy) \\ &= cb \int_0^a y^2 dy = \frac{1}{3} a^3 bc.\end{aligned}$$

Significance

For a non-constant electric field, the integral method is required.

Check Your Understanding If the electric field in the example above is $E^{\vec{r}} = mx\hat{k}$, what is the flux through the rectangular area?