## Conservation of Linear Momentum

Recall Newton's third law: When two objects of masses $m_1$ and $m_2$ interact (meaning that they apply forces on each other), the force that object 2 applies to object 1 is equal in magnitude and opposite in direction to the force that object 1 applies on object 2. Let:

- $\vec{F}_{21}$ = the force on $m_1$ from $m_2$
- $\vec{F}_{12}$ = the force on $m_2$ from $m_1$

Then, in symbols, Newton's third law says

$$\begin{aligned} \vec{\mathbf{F}}_{21} &= -\vec{\mathbf{F}}_{12} \\ m_1 \vec{\mathbf{a}}_{\mathbf{1}} &= -m_2 \vec{\mathbf{a}}_{\mathbf{2}}. \end{aligned}$$

9.10

(Recall that these two forces do not cancel because they are applied to different objects. $F_{21}$ causes $m_1$ to accelerate, and $F_{12}$ causes $m_2$ to accelerate.)

Although the magnitudes of the forces on the objects are the same, the accelerations are not, simply because the masses (in general) are different. Therefore, the changes in velocity of each object are different:

$$\frac{d\vec{v}_1}{dt} \neq \frac{d\vec{v}_2}{dt}.$$

However, the products of the mass and the change of velocity *are* equal (in magnitude):

$$m_1 \frac{d\vec{v}_1}{dt} = -m_2 \frac{d\vec{v}_2}{dt}.$$

9.11

It's a good idea, at this point, to make sure you're clear on the physical meaning of the derivatives in Equation 9.3. Because of the interaction, each object ends up getting its velocity changed, by an amount *dv*. Furthermore, the interaction occurs over a time interval *dt*, which means that the change of velocities also occurs over *dt*. This time interval is the same for each object.

Let's assume, for the moment, that the masses of the objects do not change during the interaction. (We'll relax this restriction later.) In that case, we can pull the masses inside the derivatives:

$$\frac{d}{dt}\left(m_1 \vec{v}_1\right) = -\frac{d}{dt}\left(m_2 \vec{v}_2\right)$$

9.12

and thus

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}.$$

9.13

This says that *the rate at which momentum changes is the same for both objects.* The masses are different, and the changes of velocity are different, but the rate of change of the product of $m$ and $\vec{v}$ are the same.

Physically, this means that during the interaction of the two objects ($m_1$ *and* $m_2$), both objects have their momentum changed; but those changes are identical in magnitude, though opposite in sign. For example, the momentum of object 1 might increase, which means that the momentum of object 2 decreases by exactly the same amount.

In light of this, let's re-write Equation 9.12 in a more suggestive form:

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0.$$

9.14

This says that during the interaction, although object 1's momentum changes, and object 2's momentum also changes, these two changes cancel each other out, so that the total change of momentum of the two objects together is zero.

Since the total combined momentum of the two objects together never changes, then we could write

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

9.15

from which it follows that

$$\vec{p}_1 + \vec{p}_2 = constant.$$

9.16

As shown in Figure 9.14, the total momentum of the system before and after the collision remains the same.

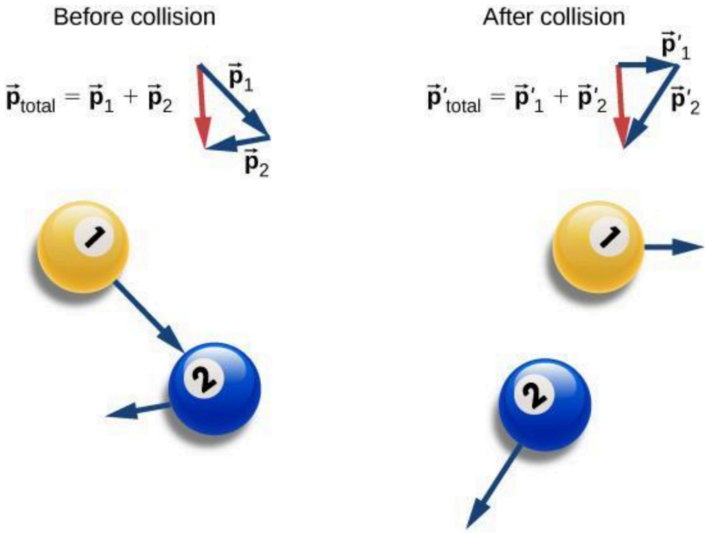



Figure 9.14 Before the collision, the two billiard balls travel with momenta $\vec{p}_1$ and $\vec{p}_2$. The total momentum of the system is the sum of these, as shown by the red vector labeled $\vec{p}_{total}$ on the left. After the collision, the two billiard balls travel with different momenta $\vec{p}'_1$ and $\vec{p}'_2$. The total momentum, however, has not changed, as shown by the red vector arrow $\vec{p}'_{total}$ on the right.

Generalizing this result to *N* objects, we obtain

$$\begin{aligned} \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_3 + \cdots + \vec{\mathbf{p}}_N &= \text{constant} \\ \sum_{j=1}^{N} \vec{\mathbf{p}}_j &= \text{constant}. \end{aligned}$$

9.17

Equation 9.17 is the definition of the total (or net) momentum of a system of *N* interacting objects, along with the statement that the total momentum of a system of objects is constant in time—or better, is conserved.

**Conservation Laws**

If the value of a physical quantity is constant in time, we say that the quantity is conserved.

### Requirements for Momentum Conservation

There is a complication, however. A system must meet two requirements for its momentum to be conserved:

1. *The mass of the system must remain constant during the interaction.*
   As the objects interact (apply forces on each other), they may *transfer* mass from

one to another; but any mass one object gains is balanced by the loss of that mass from another. The total mass of the system of objects, therefore, remains unchanged as time passes:

$$\left[\frac{dm}{dt}\right]_{system} = 0.$$

2. *The net external force on the system must be zero.*
 As the objects collide, or explode, and move around, they exert forces on each other. However, all of these forces are internal to the system, and thus each of these internal forces is balanced by another internal force that is equal in magnitude and opposite in sign. As a result, the change in momentum caused by each internal force is cancelled by another momentum change that is equal in magnitude and opposite in direction. Therefore, internal forces cannot change the total momentum of a system because the changes sum to zero. However, if there is some external force that acts on all of the objects (gravity, for example, or friction), then this force changes the momentum of the system as a whole; that is to say, the momentum of the system is changed by the external force. Thus, for the momentum of the system to be conserved, we must have

$$\vec{F}_{ext} = \vec{0}.$$

A system of objects that meets these two requirements is said to be a closed system (also called an isolated system). Thus, the more compact way to express this is shown below.

### Law of Conservation of Momentum

The total momentum of a closed system is conserved:

$$\sum_{j=1}^{N} \vec{p}_j = constant.$$

This statement is called the Law of Conservation of Momentum. Along with the conservation of energy, it is one of the foundations upon which all of physics stands. All our experimental evidence supports this statement: from the motions of galactic clusters to the quarks that make up the proton and the neutron, and at every scale in between. *In a closed system, the total momentum never changes.*

Note that there absolutely *can* be external forces acting on the system; but for the system's momentum to remain constant, these external forces have to cancel, so that the *net* external force is zero. Billiard balls on a table all have a weight force acting on them, but the weights are balanced (canceled) by the normal forces, so there is no *net* force.

### The Meaning of 'System'

A system (mechanical) is the collection of objects in whose motion (kinematics and dynamics) you are interested. If you are analyzing the bounce of a ball on the ground, you are probably only interested in the motion of the ball, and not of Earth; thus, the ball is your system. If you are analyzing a car crash, the two cars together compose your system (Figure 9.15).

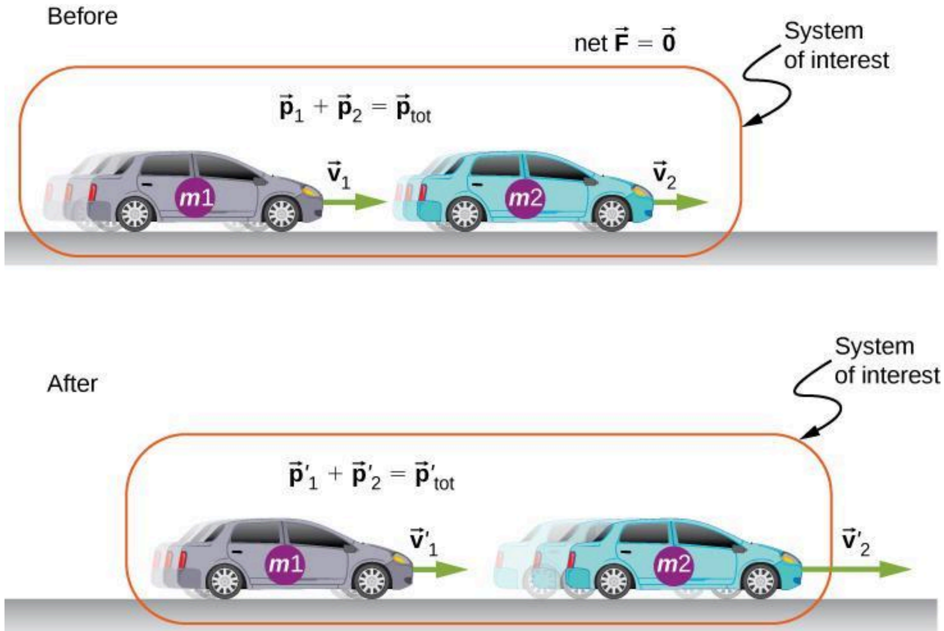



Figure 9.15 The two cars together form the system that is to be analyzed. It is important to remember that the contents (the mass) of the system do not change before, during, or after the objects in the system interact.

**Problem-Solving Strategy**

*Conservation of Momentum*

Using conservation of momentum requires four basic steps. The first step is crucial:

1. Identify a closed system (total mass is constant, no net external force acts on the system).
2. Write down an expression representing the total momentum of the system before the “event” (explosion or collision).
3. Write down an expression representing the total momentum of the system after the “event.”
4. Set these two expressions equal to each other, and solve this equation for the desired quantity.

**Example 9.6**

*Colliding Carts*

Two carts in a physics lab roll on a level track, with negligible friction. These carts have small magnets at their ends, so that when they collide, they stick together (Figure 9.16). The first cart has a mass of 675 grams and is rolling at 0.75 m/s to the right; the second has a mass of 500 grams and is rolling at 1.33 m/s, also to the right. After the collision, what is the velocity of the two joined carts?

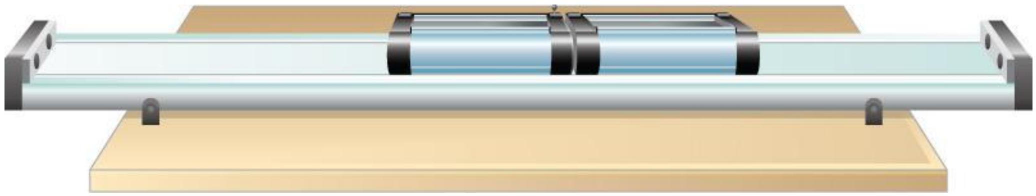


Figure 9.16 Two lab carts collide and stick together after the collision.

*Strategy*

We have a collision. We're given masses and initial velocities; we're asked for the final velocity. This all suggests using conservation of momentum as a method of solution. However, we can only use it if we have a closed system. So we need to be sure that the system we choose has no net external force on it, and that its mass is not changed by the collision.

Defining the system to be the two carts meets the requirements for a closed system: The combined mass of the two carts certainly doesn't change, and while the carts definitely exert forces on each other, those forces are internal to the system, so they do not change the momentum of the system as a whole. In the vertical direction, the weights of the carts are canceled by the normal forces on the carts from the track.

*Solution*

Conservation of momentum is

$$\vec{p}_f = \vec{p}_i.$$

Define the direction of their initial velocity vectors to be the +*x*-direction. The initial momentum is then

$$\vec{p}_i = m_1 v_1 \hat{i} + m_2 v_2 \hat{i}.$$

The final momentum of the now-linked carts is

$$\vec{p}_f = \left(m_1 + m_2\right)\vec{v}_f.$$

Equating:

$$\begin{aligned}(m_1 + m_2)\vec{\mathbf{v}}_{\mathbf{f}} &= m_1 v_1 \hat{\mathbf{i}} + m_2 v_2 \hat{\mathbf{i}} \\ \vec{\mathbf{v}}_{\mathbf{f}} &= \left(\tfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right)\hat{\mathbf{i}}.\end{aligned}$$

Substituting the given numbers:

$$\begin{aligned}\vec{\mathbf{v}}_{\mathbf{f}} &= \left[\tfrac{(0.675\,\mathrm{kg})(0.75\,\mathrm{m/s}) + (0.5\,\mathrm{kg})(1.33\,\mathrm{m/s})}{1.175\,\mathrm{kg}}\right]\hat{\mathbf{i}} \\ &= (0.997\,\mathrm{m/s})\hat{\mathbf{i}}.\end{aligned}$$

*Significance*

The principles that apply here to two laboratory carts apply identically to all objects of whatever type or size. Even for photons, the concepts of momentum and conservation of momentum are still crucially important even at that scale. (Since they are massless, the momentum of a photon is defined very differently from the momentum of ordinary objects. You will learn about this when you study quantum physics.)

**Check Your Understanding 9.3**

Suppose the second, smaller cart had been initially moving to the left. What would the sign of the final velocity have been in this case?

**Example 9.7**

*A Bouncing Superball*

A superball of mass 0.25 kg is dropped from rest from a height of $h = 1.50\ m$ above the floor. It bounces with no loss of energy and returns to its initial height (Figure 9.17).

a. What is the superball's change of momentum during its bounce on the floor?
b. What was Earth's change of momentum due to the ball colliding with the floor?
c. What was Earth's change of velocity as a result of this collision?

(This example shows that you have to be careful about defining your system.)

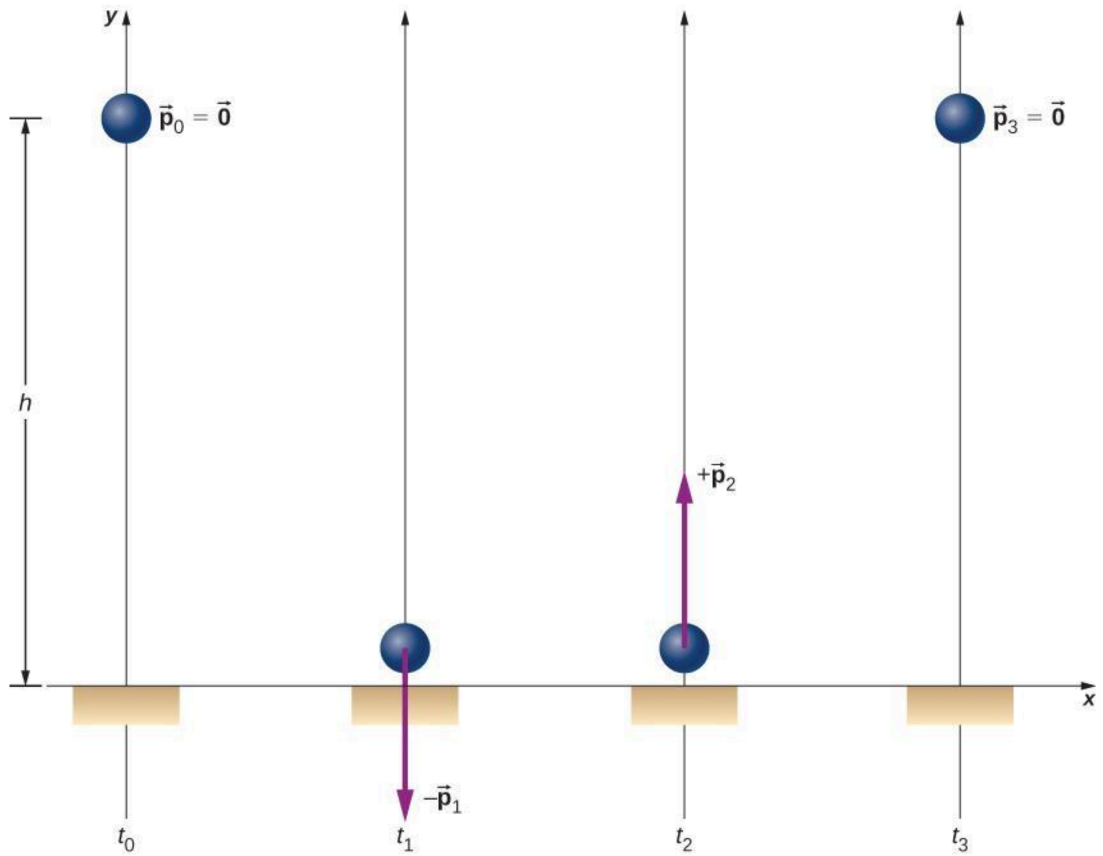



Figure 9.17 A superball is dropped to the floor ($t_0$), hits the floor ($t_1$), bounces ($t_2$), and returns to its initial height ($t_3$).

*Strategy*

Since we are asked only about the ball's change of momentum, we define our system to be the ball. But this is clearly not a closed system; gravity applies a downward force on the ball while it is falling, and the normal force from the floor applies a force during the bounce. Thus, we cannot use conservation of momentum as a strategy. Instead, we simply determine the ball's momentum just before it collides with the floor and just after, and calculate the difference. We have the ball's mass, so we need its velocities.

*Solution*

a. Since this is a one-dimensional problem, we use the scalar form of the equations. Let:

- $p_0$ = the magnitude of the ball's momentum at time $t_0$, the moment it was released; since it was dropped from rest, this is zero.
- $p_1$ = the magnitude of the ball's momentum at time $t_1$, the instant just before it hits the floor.

- $p_2$ = the magnitude of the ball's momentum at time $t_2$, just after it loses contact with the floor after the bounce.

The ball's change of momentum is

$$\begin{aligned}\Delta \overrightarrow{\mathbf{p}} &= \overrightarrow{\mathbf{p}}_2 - \overrightarrow{\mathbf{p}}_1 \\ &= p_2\hat{\mathbf{j}} - \left(-p_1\hat{\mathbf{j}}\right) \\ &= (p_2 + p_1)\hat{\mathbf{j}}.\end{aligned}$$

Its velocity just before it hits the floor can be determined from either conservation of energy or kinematics. We use kinematics here; you should re-solve it using conservation of energy and confirm you get the same result.

We want the velocity just before it hits the ground (at time $t_1$). We know its initial velocity $v_0 = 0$ (at time $t_0$), the height it falls, and its acceleration; we don't know the fall time. We could calculate that, but instead we use

$$\vec{v}_1 = -\hat{j}\sqrt{2gy} = -5.4\ m/s\hat{j}.$$

Thus the ball has a momentum of

$$\begin{aligned}\overrightarrow{\mathbf{p}}_1 &= -(0.25\ \text{kg})\left(-5.4\ \text{m/s}\hat{\mathbf{j}}\right) \\ &= -(1.4\ \text{kg}\cdot\text{m/s})\hat{\mathbf{j}}.\end{aligned}$$

We don't have an easy way to calculate the momentum after the bounce. Instead, we reason from the symmetry of the situation.

Before the bounce, the ball starts with zero velocity and falls 1.50 m under the influence of gravity, achieving some amount of momentum just before it hits the ground. On the return trip (after the bounce), it starts with some amount of momentum, rises the same 1.50 m it fell, and ends with zero velocity. Thus, the motion after the bounce was the mirror image of the motion before the bounce. From this symmetry, it must be true that the ball's momentum after the bounce must be equal and opposite to its momentum before the bounce. (This is a subtle but crucial argument; make sure you understand it before you go on.)

Therefore,

$$\vec{p}_2 = -\vec{p}_1 = +(1.4\ kg\cdot m/s)\hat{j}.$$

Thus, the ball's change of momentum during the bounce is

$$\begin{aligned}\Delta \overrightarrow{\mathbf{p}} &= \overrightarrow{\mathbf{p}}_2 - \overrightarrow{\mathbf{p}}_1 \\ &= (1.4\ \text{kg}\cdot\text{m/s})\hat{\mathbf{j}} - (-1.4\ \text{kg}\cdot\text{m/s})\hat{\mathbf{j}} \\ &= +(2.8\ \text{kg}\cdot\text{m/s})\hat{\mathbf{j}}.\end{aligned}$$

b. What was Earth's change of momentum due to the ball colliding with the floor?

Your instinctive response may well have been either "zero; the Earth is just too massive for that tiny ball to have affected it" or possibly, "more than zero, but utterly negligible." But no—if we re-define our system to be the Superball + Earth, then this system is closed (neglecting the gravitational pulls of the Sun, the Moon, and the other planets in the solar system), and therefore the total change of momentum of this new system must be zero. Therefore, Earth's change of momentum is exactly the same magnitude:

$$\Delta \vec{p}_{Earth} = -2.8\, kg \cdot m/s\, \hat{j}.$$

c. What was Earth's change of velocity as a result of this collision?

This is where your instinctive feeling is probably correct:

$$\begin{aligned} \Delta \vec{\mathbf{v}}_{\text{Earth}} &= \frac{\Delta \vec{\mathbf{p}}_{\text{Earth}}}{M_{\text{Earth}}} \\ &= -\frac{2.8\,\text{kg}\cdot\text{m/s}}{5.97\times 10^{24}\,\text{kg}}\hat{\mathbf{j}} \\ &= -\left(4.7 \times 10^{-25}\,\text{m/s}\right)\hat{\mathbf{j}}. \end{aligned}$$

This change of Earth's velocity *is* utterly negligible.

*Significance*

It is important to realize that the answer to part (c) is not a velocity; it is a change of velocity, which is a very different thing. Nevertheless, to give you a feel for just how small that change of velocity is, suppose you were moving with a velocity of $4.7 \times 10^{-25}\, m/s$. At this speed, it would take you about 7 million years to travel a distance equal to the diameter of a hydrogen atom.

**Check Your Understanding 9.4**

Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)?

Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)?

**Example 9.8**

*Ice Hockey 1*

Two hockey pucks of identical mass are on a flat, horizontal ice hockey rink. The red puck is motionless; the blue puck is moving at 2.5 m/s to the left (Figure 9.18). It collides with the motionless red puck. The pucks have a mass of 15 g. After the collision, the red puck is moving at 2.5 m/s, to the left. What is the final velocity of the blue puck?

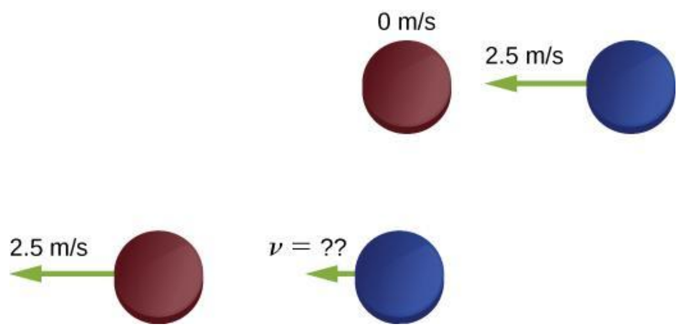



Figure 9.18 Two identical hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

*Strategy*

We're told that we have two colliding objects, we're told the masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy. Define the system to be the two pucks; there's no friction, so we have a closed system.

Before you look at the solution, what do you think the answer will be?

The blue puck final velocity will be:

- zero
- 2.5 m/s to the left
- 2.5 m/s to the right
- 1.25 m/s to the left
- 1.25 m/s to the right
- something else

*Solution*

Define the +*x*-direction to point to the right. Conservation of momentum then reads

$$\begin{aligned} \vec{\mathbf{p}}_{\mathbf{f}} &= \vec{\mathbf{p}}_{\mathbf{i}} \\ mv_{\mathrm{r_f}}\hat{\mathbf{i}} + mv_{\mathrm{b_f}}\hat{\mathbf{i}} &= mv_{\mathrm{r_i}}\hat{\mathbf{i}} - mv_{\mathrm{b_i}}\hat{\mathbf{i}}. \end{aligned}$$

Before the collision, the momentum of the system is entirely and only in the blue puck. Thus,

$$\begin{aligned} mv_{\mathrm{r_f}}\hat{\mathbf{i}} + mv_{\mathrm{b_f}}\hat{\mathbf{i}} &= -mv_{\mathrm{b_i}}\hat{\mathbf{i}} \\ v_{\mathrm{r_f}}\hat{\mathbf{i}} + v_{\mathrm{b_f}}\hat{\mathbf{i}} &= -v_{\mathrm{b_i}}\hat{\mathbf{i}}. \end{aligned}$$

(Remember that the masses of the pucks are equal.) Substituting numbers:

$$\begin{aligned} -(2.5\,\text{m/s})\hat{\mathbf{i}} + \vec{\mathbf{v}}_{\text{b}_\text{f}} &= -(2.5\,\text{m/s})\hat{\mathbf{i}} \\ \vec{\mathbf{v}}_{\text{b}_\text{f}} &= 0. \end{aligned}$$

*Significance*
Evidently, the two pucks simply exchanged momentum. The blue puck transferred all of its momentum to the red puck. In fact, this is what happens in similar collision where $m_1 = m_2$.

**Check Your Understanding 9.5**
Even if there were some friction on the ice, it is still possible to use conservation of momentum to solve this problem, but you would need to impose an additional condition on the problem. What is that additional condition?

**Example 9.9**

*Landing of Philae*
On November 12, 2014, the European Space Agency successfully landed a probe named *Philae* on Comet 67P/Churyumov/Gerasimenko (Figure 9.19). During the landing, however, the probe actually landed three times, because it bounced twice. Let's calculate how much the comet's speed changed as a result of the first bounce.

Figure 9.19 An artist's rendering of *Philae* landing on a comet. (credit: modification of work by "DLR German Aerospace Center"/Flickr)

Let's define upward to be the +*y*-direction, perpendicular to the surface of the comet, and $y = 0$ to be at the surface of the comet. Here's what we know:

- The mass of Comet 67P: $M_c = 1.0 \times 10^{13}\ kg$

- The acceleration due to the comet's gravity: $\vec{a} = -\left(5.0 \times 10^{-3}\ m/s^2\right)\hat{j}$
- *Philae's* mass: $M_p = 96\ kg$
- Initial touchdown speed: $\vec{v}_1 = -\ (1.0\ m/s)\hat{j}$
- Initial upward speed due to first bounce: $\vec{v}_2 = (0.38\ m/s)\hat{j}$
- Landing impact time: $\Delta t = 1.3\ s$

*Strategy*

We're asked for how much the comet's speed changed, but we don't know much about the comet, beyond its mass and the acceleration its gravity causes. However, we *are* told that the *Philae* lander collides with (lands on) the comet, and bounces off of it. A collision suggests momentum as a strategy for solving this problem.

If we define a system that consists of both *Philae* and Comet 67/P, then there is no net external force on this system, and thus the momentum of this system is conserved. (We'll neglect the gravitational force of the sun.) Thus, if we calculate the change of momentum of the lander, we automatically have the change of momentum of the comet. Also, the comet's change of velocity is directly related to its change of momentum as a result of the lander "colliding" with it.

*Solution*

Let $\vec{p}_1$ be *Philae's* momentum at the moment just before touchdown, and $\vec{p}_2$ be its momentum just after the first bounce. Then its momentum just before landing was

$$\vec{p}_1 = M_p\vec{v}_1 = (96\ kg)\left(-\ 1.0\ m/s\hat{j}\right) = -\ (96\ kg\cdot m/s)\hat{j}$$

and just after was

$$\vec{p}_2 = M_p\vec{v}_2 = (96\ kg)\left(+\ 0.38\ m/s\hat{j}\right) = (36.5\ kg\cdot m/s)\hat{j}.$$

Therefore, the lander's change of momentum during the first bounce is

$$\begin{aligned}\Delta\vec{\mathbf{p}} &= \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \\ &= (36.5\,\mathrm{kg}\cdot\mathrm{m/s})\hat{\mathbf{j}} - \left(-96.0\,\mathrm{kg}\cdot\mathrm{m/s}\hat{\mathbf{j}}\right) = (133\,\mathrm{kg}\cdot\mathrm{m/s})\hat{\mathbf{j}}\end{aligned}$$

Notice how important it is to include the negative sign of the initial momentum.

Now for the comet. Since momentum of the system must be conserved, the *comet's* momentum changed by exactly the negative of this:

$$\Delta\vec{p}_c = -\ \Delta\vec{p} = -\ (133\ kg\cdot m/s)\hat{j}.$$

Therefore, its change of velocity is

$$\Delta \vec{v}_c = \frac{\Delta \vec{p}_c}{M_c} = \frac{-(133\ kg{\cdot}m/s)\hat{j}}{1.0 \times 10^{13}\ kg} = -\left(1.33 \times 10^{-11}\ m/s\right)\hat{j}.$$

*Significance*

This is a very small change in velocity, about a thousandth of a billionth of a meter per second. Crucially, however, it is *not* zero.

**Check Your Understanding 9.6**

The changes of momentum for *Philae* and for Comet 67/P were equal (in magnitude). Were the impulses experienced by *Philae* and the comet equal? How about the forces? How about the changes of kinetic energies?

## Types of Collisions

Although momentum is conserved in all interactions, not all interactions (collisions or explosions) are the same. The possibilities include:

- A single object can explode into multiple objects (explosions).
- Multiple objects can collide and bounce off each other, called an elastic collision, resulting in the same kinetic energy of the system before and after the collision.
- Multiple objects can collide and the system loses kinetic energy, called an inelastic collision. One such case is where the two objects stick together, forming a single object.

It's useful, therefore, to categorize different types of interactions, according to how the interacting objects move before and after the interaction.

### Explosions

The first possibility is that a single object may break apart into two or more pieces. An example of this is a firecracker, or a bow and arrow, or a rocket rising through the air toward space. These can be difficult to analyze if the number of fragments after the collision is more than about three or four; but nevertheless, the total momentum of the system before and after the explosion is identical.

Note that if the object is initially motionless, then the system (which is just the object) has no momentum and no kinetic energy. After the explosion, the net momentum of all the pieces of the object must sum to zero (since the momentum of this closed system cannot change). However, the system *will* have a great deal of kinetic energy after the explosion, although it had none before. Thus, we see that, although the momentum of the system is conserved in an explosion, the kinetic energy of the system most definitely is not; it increases. This interaction—one object becoming many, with an increase of kinetic energy of the system—is called an explosion.

Where does the energy come from? Does conservation of energy still hold? Yes; some form of potential energy is converted to kinetic energy. In the case of gunpowder burning and pushing out a bullet, chemical potential energy is converted to kinetic energy of the bullet, and of the recoiling gun. For a bow and arrow, it is elastic potential energy in the bowstring.

### Inelastic

The second possibility is the reverse: that two or more objects collide with each other and stick together, thus (after the collision) forming one single composite object. The total mass of this composite object is the sum of the masses of the original objects, and the new single object moves with a velocity dictated by the conservation of momentum. However, it turns out again that, although the total momentum of the system of objects remains constant, the kinetic energy doesn't; but this time, the kinetic energy decreases. This type of collision is called inelastic.

Any collision where the objects stick together will result in the maximum loss of kinetic energy (i.e., $K_f$ will be a minimum).

Such a collision is called perfectly inelastic. In the extreme case, multiple objects collide, stick together, and remain motionless after the collision. Since the objects are all motionless after the collision, the final kinetic energy is also zero; therefore, the loss of kinetic energy is a maximum.

- If $0 < K_f < K_i$, the collision is inelastic.
- If $K_f$ is the lowest energy, or the energy lost by both objects is the most, the collision is perfectly inelastic (objects stick together).
- If $K_f = K_i$, the collision is elastic.

### Elastic

The extreme case on the other end is if two or more objects approach each other, collide, and bounce off each other, moving away from each other at the same relative speed at which they approached each other. In this case, the total kinetic energy of the system is conserved. Such an interaction is called elastic.

### Problem-Solving Strategy

*Collisions*

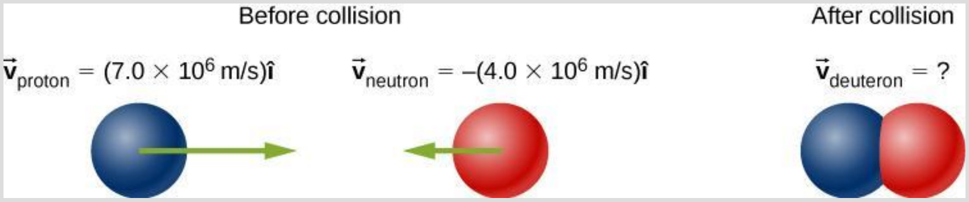
A closed system always conserves momentum; it might also conserve kinetic energy, but very often it doesn't. Energy-momentum problems confined to a plane (as ours are) usually have two unknowns. Generally, this approach works well:

1. Define a closed system.
2. Write down the expression for conservation of momentum.
3. If kinetic energy is conserved, write down the expression for conservation of kinetic energy; if not, write down the expression for the change of kinetic energy.
4. You now have two equations in two unknowns, which you solve by standard methods.

### Example 9.10

*Formation of a Deuteron*

A proton (mass $1.67 \times 10^{-27}\ kg$) collides with a neutron (with essentially the same mass as the proton) to form a particle called a *deuteron*. What is the velocity of the deuteron if it is formed from a proton moving with velocity $7.0 \times 10^{6}\ m/s$ to the left and a neutron moving with velocity $4.0 \times 10^6\ m/s$ to the right?




*Strategy*

Define the system to be the two particles. This is a collision, so we should first identify what kind. Since we are told the two particles form a single particle after the collision, this means that the collision is perfectly inelastic. Thus, kinetic energy is not conserved, but momentum is. Thus, we use conservation of momentum to determine the final velocity of the system.

*Solution*

Treat the two particles as having identical masses *M*. Use the subscripts p, n, and d for proton, neutron, and deuteron, respectively. This is a one-dimensional problem, so we have

$$Mv_p - Mv_n = 2Mv_d.$$

The masses divide out:

$$\begin{aligned} v_{\text{p}} - v_{\text{n}} &= 2v_{\text{d}} \\ 7.0 \times 10^6\ \text{m/s} - 4.0 \times 10^6\ \text{m/s} &= 2v_{\text{d}} \\ v_{\text{d}} &= 1.5 \times 10^6\ \text{m/s}. \end{aligned}$$

The velocity is thus $\vec{v}_d = \left(1.5 \times 10^6\ m/s\right)\hat{i}$.

*Significance*

This is essentially how particle colliders like the Large Hadron Collider work: They accelerate particles up to very high speeds (large momenta), but in opposite directions. This maximizes the creation of so-called “daughter particles.”

**Example 9.11**

*Ice Hockey 2*

(This is a variation of an earlier example.)

Two ice hockey pucks of different masses are on a flat, horizontal hockey rink. The red puck has a mass of 15 grams, and is motionless; the blue puck has a mass of 12 grams, and is moving at 2.5 m/s to the left. It collides with the motionless red puck (Figure 9.20). If the collision is perfectly elastic, what are the final velocities of the two pucks?

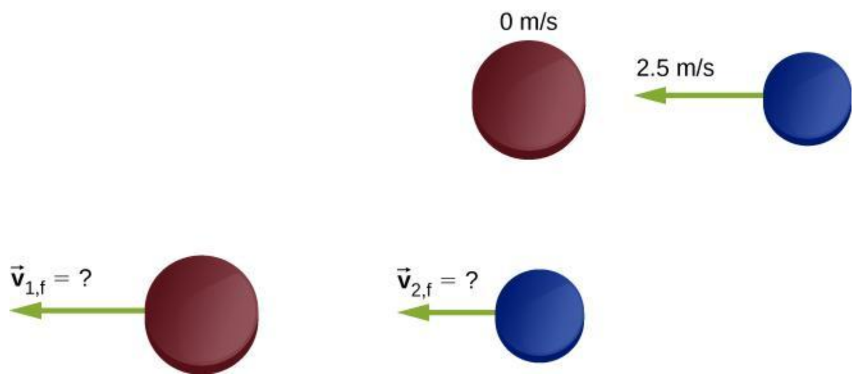



Figure 9.20 Two different hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

*Strategy*

We're told that we have two colliding objects, and we're told their masses and initial velocities; we're asked for both final velocities. Conservation of momentum seems like a good strategy; define the system to be the two pucks. There is no friction, so we have a closed system. We have two unknowns (the two final velocities), but only one equation. The comment about the collision being perfectly elastic is the clue; it suggests that kinetic energy is also conserved in this collision. That gives us our second equation.

The initial momentum and initial kinetic energy of the system resides entirely and only in the second puck (the blue one); the collision transfers some of this momentum and energy to the first puck.

*Solution*

Conservation of momentum, in this case, reads

$$\begin{aligned} p_{\text{i}} &= p_{\text{f}} \\ m_2 v_{2,\text{i}} &= m_1 v_{1,\text{f}} + m_2 v_{2,\text{f}}. \end{aligned}$$

Conservation of kinetic energy reads

$$\begin{aligned} K_{\text{i}} &= K_{\text{f}} \\ \frac{1}{2} m_2 v_{2,\text{i}}^2 &= \frac{1}{2} m_1 v_{1,\text{f}}^2 + \frac{1}{2} m_2 v_{2,\text{f}}^2. \end{aligned}$$

There are our two equations in two unknowns. The algebra is tedious but not terribly difficult; you definitely should work it through. The solution is

$$\begin{aligned} v_{1,\text{f}} &= \frac{(m_1 - m_2)v_{1,\text{i}} + 2m_2 v_{2,\text{i}}}{m_1 + m_2} \\ v_{2\text{f}} &= \frac{(m_2 - m_1)v_{2,\text{i}} + 2m_1 v_{1,\text{i}}}{m_1 + m_2}. \end{aligned}$$

Substituting with the given numbers where a positive direction is to the left, we obtain

$$v_{1,f} = 2.22\,\frac{m}{s}$$
$$v_{2,f} = -0.28\,\frac{m}{s}.$$

*Significance*

Notice that after the collision, the blue puck is moving to the right; its direction of motion was reversed. The red puck is now moving to the left.

**Check Your Understanding 9.7**

There is a possible mathematical second to the system of equations solved in this example (because the energy equation is quadratic): $v_{1,f} = 0, v_{2,f} = -2.5\ m/s$. This solution is unacceptable on physical grounds; what's wrong with it?

**Example 9.12**

*Thor vs. Iron Man*

The 2012 movie "The Avengers" has a scene where Iron Man and Thor fight. At the beginning of the fight, Thor throws his hammer at Iron Man, hitting him and throwing him slightly up into the air and against a small tree, which breaks. From the video, Iron Man is standing still when the hammer hits him. The distance between Thor and Iron Man is approximately 10 m, and the hammer takes about 1 s to reach Iron Man after Thor releases it. The tree is about 2 m behind Iron Man, which he hits in about 0.75 s. Also from the video, Iron Man's trajectory to the tree is very close to horizontal. Assuming Iron Man's total mass is 200 kg:

a. Estimate the mass of Thor's hammer
b. Estimate how much kinetic energy was lost in this collision

*Strategy*

After the collision, Thor's hammer is in contact with Iron Man for the entire time, so this is a perfectly inelastic collision. Thus, with the correct choice of a closed system, we expect momentum is conserved, but not kinetic energy. We use the given numbers to estimate the initial momentum, the initial kinetic energy, and the final kinetic energy. Because this is a one-dimensional problem, we can go directly to the scalar form of the equations.

*Solution*

a. First, we posit conservation of momentum. For that, we need a closed system. The choice here is the system (hammer + Iron Man), from the time of collision to the moment just before Iron Man and the hammer hit the tree. Let:

   - $M_H$ = mass of the hammer
   - $M_I$ = mass of Iron Man
   - $v_H$ = velocity of the hammer before hitting Iron Man

– $v$ = combined velocity of Iron Man + hammer after the collision

Again, Iron Man's initial velocity was zero. Conservation of momentum here reads:

$$M_H v_H = \left(M_H + M_I\right)v.$$

We are asked to find the mass of the hammer, so we have

$$\begin{aligned} M_{\rm H}v_{\rm H} &= M_{\rm H}v + M_{\rm I}v \\ M_{\rm H}(v_{\rm H} - v) &= M_{\rm I}v \\ M_{\rm H} &= \frac{M_{\rm I}v}{v_{\rm H}-v} \\ &= \frac{(200\,{\rm kg})\left(\frac{2\,{\rm m}}{0.75\,{\rm s}}\right)}{10\,\frac{\rm m}{\rm s} - \left(\frac{2\,{\rm m}}{0.75\,{\rm s}}\right)} \\ &= 73\,{\rm kg}. \end{aligned}$$

Considering the uncertainties in our estimates, this should be expressed with just one significant figure; thus, $M_H = 7 \times 10^1\ kg$.

b. The initial kinetic energy of the system, like the initial momentum, is all in the hammer:

$$\begin{aligned} K_{\rm i} &= \tfrac{1}{2}M_{\rm H}v_{\rm H}^2 \\ &= \tfrac{1}{2}(70\,{\rm kg})(10\,{\rm m/s})^2 \\ &= 3500\,{\rm J}. \end{aligned}$$

After the collision,

$$\begin{aligned} K_{\rm f} &= \tfrac{1}{2}(M_{\rm H} + M_{\rm I})v^2 \\ &= \tfrac{1}{2}(70\,{\rm kg} + 200\,{\rm kg})(2.67\,{\rm m/s})^2 \\ &= 960\,{\rm J}. \end{aligned}$$

Thus, there was a loss of $3500\ J - 960\ J = 2540\ J$.

*Significance*

From other scenes in the movie, Thor apparently can control the hammer's velocity with his mind. It is possible, therefore, that he mentally causes the hammer to maintain its initial velocity of 10 m/s while Iron Man is being driven backward toward the tree. If so, this would represent an external force on our system, so it would not be closed. Thor's mental control of his hammer is beyond the scope of this book, however.

**Example 9.13**

*Analyzing a Car Crash*

At a stoplight, a large truck (3000 kg) collides with a motionless small car (1200 kg). The truck comes to an instantaneous stop; the car slides straight ahead, coming to a stop after sliding 10 meters. The measured coefficient of friction between the car's tires and the road was 0.62. How fast was the truck moving at the moment of impact?

*Strategy*
At first it may seem we don't have enough information to solve this problem. Although we know the initial speed of the car, we don't know the speed of the truck (indeed, that's what we're asked to find), so we don't know the initial momentum of the system. Similarly, we know the final speed of the truck, but not the speed of the car immediately after impact. The fact that the car eventually slid to a speed of zero doesn't help with the final momentum, since an external friction force caused that. Nor can we calculate an impulse, since we don't know the collision time, or the amount of time the car slid before stopping. A useful strategy is to impose a restriction on the analysis.

Suppose we define a system consisting of just the truck and the car. The momentum of this system isn't conserved, because of the friction between the car and the road. But if we *could* find the speed of the car the instant after impact—before friction had any measurable effect on the car—then we could consider the momentum of the system to be conserved, with that restriction.

Can we find the final speed of the car? Yes; we invoke the work-energy theorem.

*Solution*
First, define some variables. Let:

- $M_c$ $and$ $M_T$ be the masses of the car and truck, respectively
- $v_{T,i}$ $andv_{T,f}$ be the velocities of the truck before and after the collision, respectively
- $v_{c,i}$ $andv_{c,f}$ Z be the velocities of the car before and after the collision, respectively
- $K_i$ $and$ $K_f$ be the kinetic energies of the car immediately after the collision, and after the car has stopped sliding (so $K_f = 0$).
- *d* be the distance the car slides after the collision before eventually coming to a stop.

Since we actually want the initial speed of the truck, and since the truck is not part of the work-energy calculation, let's start with conservation of momentum. For the car + truck system, conservation of momentum reads

$$p_{\mathrm{i}} = p_{\mathrm{f}}$$
$$M_{\mathrm{c}}v_{\mathrm{c,i}} + M_{\mathrm{T}}v_{\mathrm{T,i}} = M_{\mathrm{c}}v_{\mathrm{c,f}} + M_{\mathrm{T}}v_{\mathrm{T,f}}.$$

Since the car's initial velocity was zero, as was the truck's final velocity, this simplifies to

$$v_{T,i} = \frac{M_c}{M_T}v_{c,f}.$$

So now we need the car's speed immediately after impact. Recall that

$$W = \Delta K$$

where

$$\begin{aligned}\Delta K &= K_f - K_i \\ &= 0 - \frac{1}{2} M_c v_{c,f}^2.\end{aligned}$$

Also,

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta.$$

The work is done over the distance the car slides, which we've called *d*. Equating:

$$Fd\cos\theta = -\frac{1}{2} M_c v_{c,f}^2.$$

Friction is the force on the car that does the work to stop the sliding. With a level road, the friction force is

$$F = \mu_k M_c g.$$

Since the angle between the directions of the friction force vector and the displacement *d* is 180°, and $\cos(180°) = -1$, we have

$$-\left(\mu_k M_c g\right)d = -\frac{1}{2} M_c v_{c,f}^2$$

(Notice that the car's mass divides out; evidently the mass of the car doesn't matter.)

Solving for the car's speed immediately after the collision gives

$$v_{c,f} = \sqrt{2\mu_k g d}.$$

Substituting the given numbers:

$$\begin{aligned}v_{c,f} &= \sqrt{2(0.62)\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)(10\ \text{m})}. \\ &= 11.0\ \text{m/s}.\end{aligned}$$

Now we can calculate the initial speed of the truck:

$$v_{T,i} = \left(\frac{1200\ kg}{3000\ kg}\right)\left(11.0\ \frac{m}{s}\right) = 4.4\ m/s.$$

*Significance*

This is an example of the type of analysis done by investigators of major car accidents. A great deal of legal and financial consequences depend on an accurate analysis and calculation of momentum and energy.

**Check Your Understanding 9.8**

Suppose there had been no friction (the collision happened on ice); that would make $\mu_k$ zero, and thus $v_{c,f} = \sqrt{2\mu_k g d} = 0$, which is obviously wrong. What is the mistake in this conclusion?

### Subatomic Collisions and Momentum

Conservation of momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

At the beginning of the twentieth century, there was considerable interest in, and debate about, the structure of the atom. It was known that atoms contain two types of electrically charged particles: negatively charged electrons and positively charged protons. (The existence of an electrically neutral particle was suspected, but would not be confirmed until 1932.) The question was, how were these particles arranged in the atom? Were they distributed uniformly throughout the volume of the atom (as J.J. Thomson proposed), or arranged at the corners of regular polygons (which was Gilbert Lewis' model), or rings of negative charge that surround the positively charged nucleus—rather like the planetary rings surrounding Saturn (as suggested by Hantaro Nagaoka), or something else?

The New Zealand physicist Ernest Rutherford (along with the German physicist Hans Geiger and the British physicist Ernest Marsden) performed the crucial experiment in 1909. They bombarded a thin sheet of gold foil with a beam of high-energy (that is, high-speed) alpha-particles (the nucleus of a helium atom). The alpha-particles collided with the gold atoms, and their subsequent velocities were detected and analyzed, using conservation of momentum and conservation of energy.

If the charges of the gold atoms were distributed uniformly (per Thomson), then the alpha-particles should collide with them and nearly all would be deflected through many angles, all small; the Nagaoka model would produce a similar result. If the atoms were arranged as regular polygons (Lewis), the alpha-particles would deflect at a relatively small number of angles.

What *actually* happened is that nearly *none* of the alpha-particles were deflected. Those that were, were deflected at large angles, some close to 180°—those alpha-particles reversed direction completely (Figure 9.21). None of the existing atomic models could explain this. Eventually, Rutherford developed a model of the atom that was much closer to what we now have—again, using conservation of momentum and energy as his starting point.

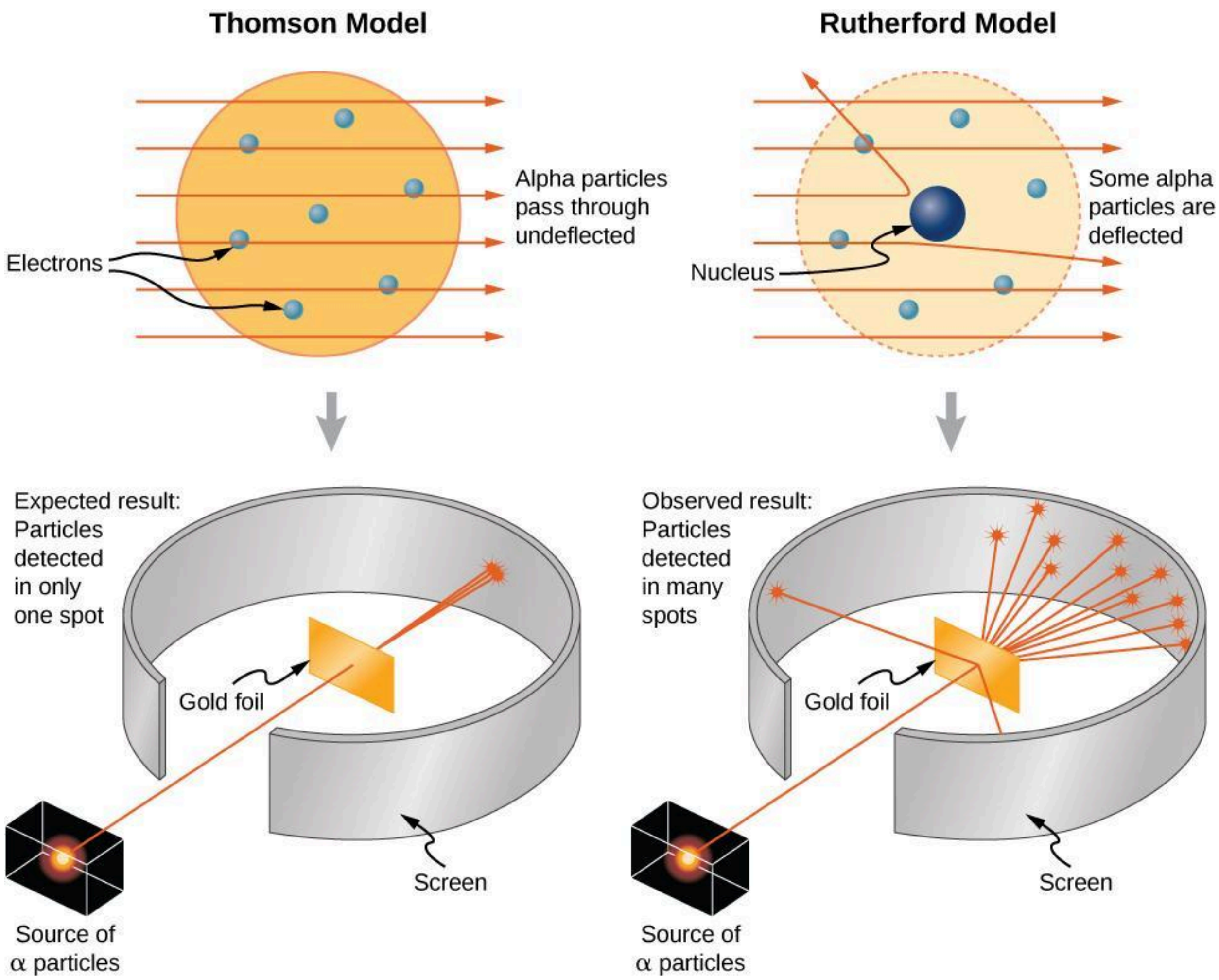



Figure 9.21 The Thomson and Rutherford models of the atom. The Thomson model predicted that nearly all of the incident alpha-particles would be scattered and at small angles. Rutherford and Geiger found that nearly none of the alpha particles were scattered, but those few that were deflected did so through very large angles. The results of Rutherford's experiments were inconsistent with the Thomson model. Rutherford used conservation of momentum and energy to develop a new, and better model of the atom—the nuclear model.